The Concentration Problem for Vector Fields.

H. Maniar[1], P. P. Mitra[1]
[1] Department of Neuroscience, Cold Spring Harbor Laboratory, NY, USA

***Abstract*— According to the uncertainty principle, a function limited to a finite interval (a time-limited or a space-limited function) cannot be strictly limited to a finite frequency (or spatial frequency) interval or region. One can however find functions that are time (or space) limited, and are *maximally* frequency limited. One way to find such functions is to define a spectral concentration, namely the fractional energy of a time-limited function within a limited frequency range. The concentration problem leads to an eigenvalue problem which provides an eigenbasis with strict time limitation and good frequency limitation. The problem is symmetric, and one may look for strictly frequency limited functions which are maximally time limited. Analogous functions may be found on the sphere, with spherical harmonics playing the role of sinusoids and angular momenta playing the role of frequencies. With appropriate transformations, it becomes apparent that the MEG inverse problem can be treated using such functions. We describe this approach using an idealised geometry where 'space' corresponds to a spherical source space, and 'frequency' corresponds to the coefficients of the magnetic field expanded in a spherical harmonic basis. Concentric spherical head models with spherically symmetric conductivity profile are often used for MEG source localization, and it is in this context that we discuss the concentration problem for vector fields on a sphere.**

***Index Terms*— Localization, Inverse Problem, Linear Estimation, Concentration Problem, Multi-Taper, Vector Fields, Vector Spherical Harmonics, Magnetoencephalography.**

I. INTRODUCTION

Concentric spherical head models with spherically symmetric conductivity profile are often used for MEG source localization. Vector spherical harmonics provide a convenient but not necessarily an efficient basis set for representing the primary currents which are spatially sparse and focal. Space-limited functions such as the primary current include many spatial frequencies, and any naive attempt to represent it using a limited spectral range (as would be obtained by expanding the measured magnetic fields in a spherical harmonic basis) will result in aliasing due to the non-negligible energy of the neglected higher spectral components, eventually resulting in a *biased* estimate. We propose that vector fields, such as the primary current, are optimally represented using a *local basis set* constructed by solving a spectral concentration problem on the sphere. In spirit the idea is similar to the time-frequency multitaper spectral analysis [2].

We begin with a brief review of the concentration problem in the time-frequency (1D) domain, followed by a review of the concentration problem for scalar fields on a sphere (2D). Finally, we discuss the concentration problem for a tangential vector field on a sphere. The concepts are easily extended to non-spherical geometries and more general vector fields. We refer the reader to [3] for a more detailed discussion in the MEG imaging context.

II. TIME-FREQUENCY DOMAIN

We begin by briefly reviewing the concentration problem for the continuous-continuous time(t)-frequency(ω) domain. A strictly band-limited signal, $f(t)$, is one whose spectra, $F(\omega)$, vanishes for frequencies $|\omega| > W$, and a strictly time-limited signal one which is non-zero over a finite interval of time. Strictly speaking, a band-limited signal cannot be simultaneously time-limited, and vice-versa. In practice, one can approximately concentrate a band-limited signal within a finite time-interval (and vice-versa), and the concentration problem provides a prescription for doing so.

One measure of *temporal concentration* of signal energy is

$$\lambda = \frac{\int_{-T}^{T} f^2(t)dt}{\int_{-\infty}^{\infty} f^2(t)dt}. \tag{1}$$

From the space of all band-limited signal, $f(t)$, the concentration problem seeks to identify those which have maximum energy within the time-interval, $|t| < T$, ie, one's which are well-concentrated. Formally, one maximizes the measure of temporal concentration (1). On substitution of the Fourier representation of $f(t)$, it can be shown that the resulting variational problem is equivalent to the integral eigenvalue equation for the spectra, $F(\omega)$, $|\omega| < W$,

$$\int_{-W}^{W} \frac{\sin T(\omega - \omega')}{\pi(\omega - \omega')} F(\omega')d\omega' = \lambda F(\omega) \tag{2}$$

The solution to which provides eigenvalues, $1 > \lambda_1 > \lambda_2 > \ldots$, and the *eigenspectra*, $F_1(\omega), F_2(\omega), \ldots$. The inverse transform of the eigenspectra provide the time-domain eigenfunctions (*tapers*), $f_1(t), f_2(t), \ldots$. The eigenvalues are a measure of temporal concentration. The important point to note is that typically only a few of the eigenvalues are significant, ie close to 1, and given by the time-bandwidth product, $2TW$. The eigenfunction, f_1, corresponding to λ_1 is the function *best* concentrated within the specified time-interval. The eigenfunction, f_2, corresponding to λ_2 is the next best concentrated function and orthogonal to f_1, and so on. The concentration problem provides a basis set, f_p, which on using a few leading eigenfunctions, an arbitrary band-limited function can also be simultaneously *nearly* well-concentrated in the time-domain.

Above we described the time-domain concentration problem. Instead, one can consider the complementary frequency-domain concentration problem of concentrating a time-limited signal in the frequency-domain. One encounters a time-domain integral equation similar to (2). It's solution directly provide time-domain eigenfunctions, which on Fourier transforming, provide the associated eigenspectra. The leading few eigenspectra associated with eigenvalues close to unity, are highly concentrated in the frequency band, $|\omega| < W$, and provide a local basis in the frequency domain to localize a time-limited signal. Note that the eigenfunctions and eigenspectra obtained by these complementary approaches coincide for $|t| < T$ and $|\omega| < W$.

In practice, the time domain is typically discrete, and one poses a discrete-continuous time-frequency concentration problem. See [2] for further details.

III. Scalar Field on a Sphere

Consider a spatio-spectral concentration problem for scalar fields on a sphere [1]. Scalar functions on a sphere can be represented by a spectral expansion using spherical harmonics, Y_{lm}, as a basis set (see Appendix),

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm} Y_{lm}(\theta,\phi) \tag{3}$$

where the expansion coefficients are given by,

$$f_{lm} = \int_{\Omega_\infty} f(\theta,\phi) Y^*_{lm}(\theta,\phi) \tag{4}$$

and Ω_∞ is the surface of the entire sphere. Here a band-limited function is one where the upper l-index in (3) is finite, say L. Theoretically, it is not possible to have a function that is simultaneously band-limited and *space-limited*.

One defines a *measure of spatial concentration* as

$$\lambda = \frac{\int_{\Omega_*} f^2 d\Omega}{\int_{\Omega_\infty} f^2 d\Omega}. \tag{5}$$

From the entire class of scalar band-limited functions on the sphere, the *spatial concentration problem* attempts to identify those functions which exhibit maximal spatial concentration within the specified region of interest, Ω_*. Formally, one maximizes the measure of spatial concentration (5). On substituting (3) (with an upper index, L), one solves,

$$\lambda = \max_{\mathbf{f}} \frac{\mathbf{f}^t \mathbf{D}^* \mathbf{f}}{\mathbf{f}^t\ \mathbf{f}}, \tag{6}$$

where the matrix,

$$D^*_{lm,l'm'} = \int_{\Omega_*} Y_{lm} Y^*_{l'm'}, \tag{7}$$

and $\mathbf{f} = [f_{00}, f_{1,-1}, f_{1,0}, f_{1,1}, \ldots]^t$. The variational problem is equivalent to an ordinary eigenvalue problem, $\mathbf{D}^*\mathbf{f} = \lambda\mathbf{f}$. The solution provides the eigenvalues, $1 > \lambda_1, \ldots, \lambda_{(L+1)^2} > 0$, and the *spectral eigenvectors*, $\mathbf{f}_1, \ldots, \mathbf{f}_{(L+1)^2}$. Substitution of these eigenvectors in (3) provides the *spatial eigenfunctions*, $f_1, \ldots, f_{(L+1)^2}$. For a sufficiently small Ω_*, only a few eigenvalues are close to unity. The associated spatial eigenfunctions are highly concentrated within Ω_*, and constitute a *local basis set*. Using this local basis set, an arbitrary band-limited function can be simultaneously well-concentrated within the region of interest, Ω_*.

Note that unlike the time-frequency case where the domain was one-dimensional, here the shape of the region of interest, Ω_* can have a bearing on the concentration solution. Above we considered the spatial concentration problem, similarly one can pose the *spectral concentration problem*, ie. seek to concentrate a space-limited function in the spectral-domain. It can be shown that the spectral & spatial concentration problems are equivalent [1].

IV. Vector Field on a Sphere

Vector spherical harmonics form a complete basis set on a sphere and can be used to express an arbitrary vector function, $\vec{f}(\theta,\phi)$, tangential to the surface of a sphere (see Appendix). For the MEG source localization problem for a concentric spherical head model one only encounters the $\vec{V}$-harmonics (see section to follow). For that reason, here we consider only tangential vector functions which are represented using $\vec{V}$-harmonics,

$$\vec{f}(\theta,\phi) = \sum_{l=1}^{L}\sum_{m=-l}^{l} f_{lm} \vec{V}_{lm}(\theta,\phi). \tag{8}$$

As before, one defines a measure of spatial concentration as

$$\lambda = \frac{\|\vec{f}\|^2_{\Omega_*}}{\|\vec{f}\|^2_{\Omega_\infty}} \tag{9}$$

where

$$\|\vec{f}\|^2_{\Omega_\alpha} = \int\int_{\Omega_\alpha} \vec{f}\cdot\vec{f}^* d\Omega. \tag{10}$$

The *spatial concentration problem* seeks to identify a the class of band-limited vector functions with best concentration properties in the specified region of interest, Ω_*. Formally, one maximizes (9). On substituting (8) in (9), we solve,

$$\lambda = \max_{\mathbf{f}} \frac{\mathbf{f}^t \Delta^* \mathbf{f}}{\mathbf{f}^t \Delta^\infty\ \mathbf{f}} \tag{11}$$

where $\mathbf{f} = [f_{00}, f_{1,-1}, f_{1,0}, f_{1,1}, \ldots]^t$, and the matrix

$$\Delta^\alpha_{lm,l'm'} = \int_{\Omega_\alpha} \vec{V}_{lm}\cdot\vec{V}^*_{l'm'} d\Omega. \tag{12}$$

The solution to (11) is equivalent to solving a generalized eigenvalue problem,

$$\Delta^*\mathbf{f} = \lambda\Delta^\infty\mathbf{f} \tag{13}$$

Noting the orthogonality relation (24), Δ^∞ is an identity matrix, the problem is reduced to, $\Delta^*\mathbf{f} = \lambda\mathbf{f}$. The solution to which provides $(L+1)^2 - 1$ *spectral eigenvectors*, $\mathbf{f}_1, \mathbf{f}_2, \ldots$. Each of these $(L+1)^2 - 1$ spectral eigenvectors, $\mathbf{f}_p$, provide *spatial eigenfunctions*, $\vec{f}_p$, via equation (8). Since the matrices above are Hermitian and positive-definite, these new eigenvectors and eigenfunctions have the following interesting properties

$$\begin{aligned} \mathbf{f}^t_m \Delta^* \mathbf{f}_n &= \lambda_m \delta_{mn} \\ \mathbf{f}^t_m \Delta^\infty \mathbf{f}_n &= \delta_{mn} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \textstyle\int_{\Omega_*} \vec{f}_m\cdot\vec{f}^*_n d\Omega &= \lambda_m\delta_{mn} \\ \textstyle\int_{\Omega_\infty} \vec{f}_m\cdot\vec{f}^*_n d\Omega &= \delta_{mn} \end{aligned} \tag{15}$$

The eigenvalues are a measure of concentration, and for sufficiently small Ω_*, only a few are close to unity. The associated eigenfunctions constitute a local basis set in the region of interest, Ω_*, and a tangential function $\vec{f}$ can be optimally represented using this local basis set.

V. Application to MEG

Consider a head model where the source and sensor spaces are concentric thin shells with radii a and b ($b > a$) respectively. Assuming a spherically symmetric conductivity profile, the magnetic leadfield is,

$$\vec{K}(\mathbf{r}',\mathbf{r}) = \frac{\mu_0}{4\pi b}\mathbf{r}' \times \nabla'(\frac{1}{R}) \qquad (16)$$

where $R = |\mathbf{r} - \mathbf{r}'|$, and μ_0 the free space magnetic permittivity. We express the leadfield as a *vector spherical harmonic* expansion, $\vec{K} = \sum_{lm} k_{lm}\vec{V}_{lm}(\Omega')Y_{lm}(\Omega)$ (see Appendix), and perform a spherical harmonic fit over the entire outer sensor space sphere, $B = \sum_{lm} B_{lm}Y_{lm}(\Omega)$. The integral equation is transformed to,

$$B_{lm} = \int_{\Omega'_\infty} \vec{V}_{lm}(\mathbf{r}') \cdot \vec{J}^{(p)}(\mathbf{r}')d\Omega' \qquad (17)$$

We neglect very low and high components to filter out the noise. In view of the form of the kernel, eqn. (16), note that not only are the radial components of the primary current *magnetically silent*, but so are the tangential $\vec{U}$-harmonic components (see Appendix) of the primary current, ie, $\vec{U}_{lm} \cdot \vec{K} = 0$.

Proceeding, we select a region of interest in the source space to be an arbitrarily oriented spherical cap, Ω_*, within which to estimate $\vec{J}^{(p)}$. The concentration problem, as noted in the previous section, is solved for to obtain the spectral eigenvectors, $\mathbf{f}$. Using these spectral eigenvectors, (17) can be linearly transformed to

$$\mathcal{B}_p = \int_{\Omega'_\infty} \vec{f}_p(\mathbf{r}') \cdot \vec{J}^{(p)}(\mathbf{r}')d\Omega' \qquad (18)$$

where $\vec{f}_p$ are the new *local basis expansions* or the spatial eigenfunctions specific to the ROI, Ω_* (see previous section). While there are a total of $(L+1)^2 - 1$ eigenfunctions, only a few concentration eigenvalues are close to unity, and the associated eigenfunctions, $\vec{f}_p$, are retained. Using the same local basis set to estimate $\vec{J}^{(p)}$, and noting the orthogonality condition, (15), the first moment estimate for current is given as, $\vec{J}^{(p)} = \sum_m \mathcal{B}_m\vec{f}_m$.

VI. Results

We present some results from the solution of the concentration problem for tangential vector fields on a sphere. MEG specific results will be presented in another presentation at this conference. Consider an axisymmetric spherical cap as the region of interest, Ω^*. Concentration angle is the polar angle of this spherical cap. For various ratio's, $\rho = \Omega^*/\Omega^\infty$, Figs. 1 & 2 show the concentration eigenvalue distribution for $L = 5$ & $L = 7$ respectively. As the ROI, Ω_*, is reduced, only a few eigenvalues are close to $\lambda_p \approx 1$, and the associated eigenfunctions, $\vec{f}_p$ provide a local basis set for approximation in Ω^*. Fig. 3 shows the highly focal nature of the new eigenfunctions, $\vec{f}$. The figure shows the magnitude of 2 leading eigenfunctions for a concentration angle of $\pi/5$. Note that the scale is in decibels (dB).

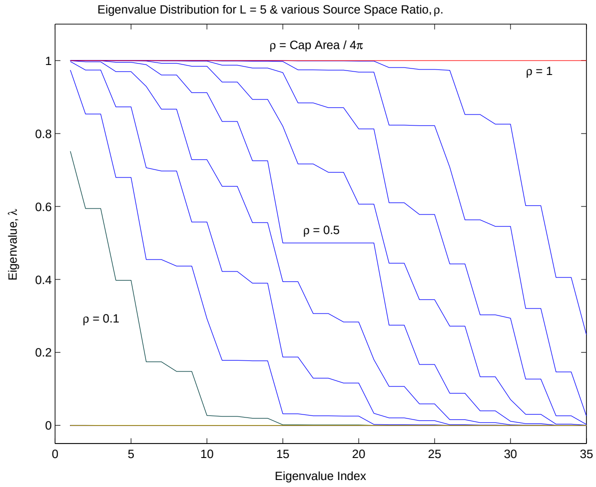

Fig. 1. Eigenvalues Distribution for the case $L = 5$.

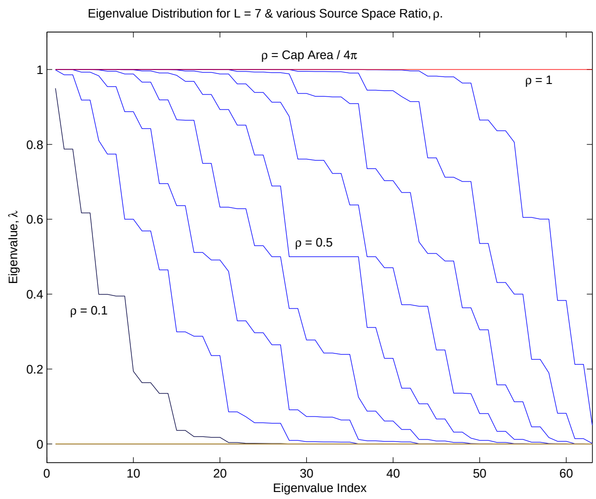

Fig. 2. Eigenvalues Distribution for the case $L = 7$.

VII. Discussion

Evoked primary current is spatially sparse and focal, and is best represented by a local basis set. The key idea is to derive these highly concentrated or localized basis sets within the region of interest by solving for a concentration problem. The ideas presented here are easily extended to non-spherical geometries and other vector fields.

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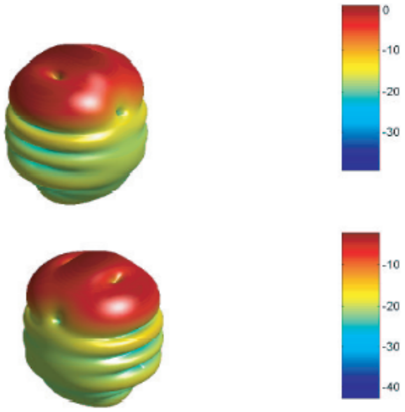

Fig. 3. The magnitude of two leading eigenfunctions for a concentration angle $\theta_0 = \pi/5$. The corresponding concentration eigenvalue is 0.8336. Note the dB scale.

Appendix A
Vector Spherical Harmonics

Vector spherical harmonics, $(\vec{U}_{lm}, \vec{V}_{lm})$, are defined as,

$$\begin{aligned} \sqrt{l(l+1)}\vec{U}_{lm} &= r\nabla_s Y_{lm} \\ \vec{V}_{lm} &= \mathbf{e}_r \times \vec{U}_{lm} \end{aligned} \tag{19}$$

where $Y_{lm} = Y_{lm}(\theta, \phi)$ is the scalar spherical harmonic,

$$Y_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta)e^{im\phi} \tag{20}$$

and ∇_s the *surface gradient* operator on a sphere of radius, r,

$$\nabla_s = \frac{1}{r}\left\{\mathbf{e}_\theta \frac{\partial}{\partial\theta} + \mathbf{e}_\phi \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\}. \tag{21}$$

It is clear from (19) that $\vec{U}_{lm} \cdot \vec{V}_{lm} = 0$, i.e. $\vec{U}_{lm} \perp \vec{V}_{lm}$. Vector spherical harmonics form a complete basis set on a sphere and an arbitrary vector function, $\vec{f}(\theta, \phi)$, tangential to the surface of a sphere can be expressed as

$$\vec{f}(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f_{lm}^u \vec{U}_{lm}(\theta, \phi) + f_{lm}^v \vec{V}_{lm}(\theta, \phi), \tag{22}$$

where the expansion coefficients, f_{lm}^u, f_{lm}^v are given by

$$\left\{ \begin{array}{c} f_{lm}^u \\ f_{lm}^v \end{array} \right\} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left\{ \begin{array}{c} \vec{U}_{lm}^*(\theta, \phi) \\ \vec{V}_{lm}^*(\theta, \phi) \end{array} \right\} \cdot \vec{f}(\theta, \phi) \sin\theta \tag{23}$$

using the orthogonality relations,

$$\left(\vec{A}_{lm}, \vec{B}_{l'm'}\right) = \delta_{\vec{A}\vec{B}} \delta_{ll'} \delta_{mm'}. \tag{24}$$

Above $\vec{A}, \vec{B}$, can be $\vec{U}_{lm}$ or $\vec{V}_{lm}$, and $(\cdot, \cdot)$ is the vector inner product,

$$(\vec{A}, \vec{B}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \;\; \vec{A}^* \cdot \vec{B} \sin\theta.$$

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